

LATTICES OF HOMOMORPHISMS AND PRO-LIE GROUPS

ARKADY G. LEIDERMAN AND MIKHAIL G. TKACHENKO¹

ABSTRACT. Early this century K. H. Hofmann and S. A. Morris introduced the class of *pro-Lie groups* which consists of projective limits of finite-dimensional Lie groups and proved that it contains all compact groups, all locally compact abelian groups, and all connected locally compact groups and is closed under the formation of products and closed subgroups. They defined a topological group G to be *almost connected* if the quotient group of G by the connected component of its identity is compact.

We show here that all almost connected pro-Lie groups as well as their continuous homomorphic images are \mathbb{R} -factorizable and ω -cellular, i.e. every family of G_δ -sets contains a countable subfamily whose union is dense in the union of the whole family. We also prove a general result which implies as a special case that if a topological group G contains a compact invariant subgroup K such that the quotient group G/K is an almost connected pro-Lie group, then G is \mathbb{R} -factorizable and ω -cellular.

Applying the aforementioned result we show that the sequential closure and the closure of an arbitrary $G_{\delta,\Sigma}$ -set in an almost connected pro-Lie group H coincide.

1. INTRODUCTION

A topological group is called a *pro-Lie group* [9, 10, 11] if it is a projective limit of finite-dimensional Lie groups. As shown in [9] the class of pro-Lie groups includes all locally compact abelian topological groups, all compact groups, all connected locally compact topological groups, and all almost connected locally compact topological groups. Further, the class of pro-Lie groups is productive and every closed subgroup of a pro-Lie group is again a pro-Lie group.

Our main objective is to study the topological properties of *almost connected* pro-Lie groups, i.e. the pro-Lie groups G such that the quotient group G/G_0 is compact, where G_0 is the connected component of G . The following theorem about the topological structure of almost connected pro-Lie groups plays a key role throughout the paper. According to [10, Corollary 8.9], an almost connected pro-Lie group is *homeomorphic* to the product $\mathbb{R}^\kappa \times K$, where \mathbb{R} is the real line with the usual topology, κ is a cardinal, and K is a compact topological group.

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A topological group G is said to be \mathbb{R} -factorizable if, for any continuous function $f: G \rightarrow \mathbb{R}$, one can find a second-countable topological group S , a continuous homomorphism $\pi: G \rightarrow S$, and a continuous function $h: S \rightarrow \mathbb{R}$ for which $f = h \circ \pi$.

Evidently, every second-countable group is \mathbb{R} -factorizable. In fact, an arbitrary topological product of second-countable topological groups is \mathbb{R} -factorizable [2, Corollary 8.1.15]. A theorem of Pontryagin, reformulated in modern terms, asserts that every compact topological group is \mathbb{R} -factorizable (see [2, Section 8.1]).

\mathbb{R} -factorizable groups were introduced in [15]. The class of \mathbb{R} -factorizable groups is sufficiently large—it contains arbitrary subgroups of Lindelöf Σ -groups, direct products of Lindelöf Σ -groups, and their dense subgroups (see [2, Section 8.1]). In particular, every subgroup of a σ -compact group is \mathbb{R} -factorizable. Nevertheless, several major questions are still open. It is not known whether the class of \mathbb{R} -factorizable groups is closed under taking direct products, or whether this class is closed under passing to continuous homomorphic images.

Delimiting the frontiers of the class of \mathbb{R} -factorizable groups is another important part of research. It is known that ω -narrow topological groups can fail to be \mathbb{R} -factorizable [2, Example 8.2.1]. Recently, E. Reznichenko and O. Sipacheva [12] obtained a considerably stronger result. They showed that there exists a separable (and hence *ccc* and ω -narrow) topological group which is not \mathbb{R} -factorizable.

Thus, the question about \mathbb{R} -factorizability of almost connected pro-Lie groups and close to them topological groups arises quite naturally.

Section 2 of the article is devoted to the study of continuous homomorphic images of the almost connected pro-Lie groups. It is shown in Theorem 2.1 that every continuous homomorphic image H of an almost connected pro-Lie group, and the Hewitt-Nachbin completion of H , are both \mathbb{R} -factorizable and ω -cellular. Further, Theorem 2.10 states that the group H is an Efimov space, and that the Hewitt-Nachbin completion of H remains to be Efimov spaces as well. All statements mentioned above fail to hold for general pro-Lie groups, without the assumption of almost connectedness (see Remark 2.3). To show that almost connected pro-Lie groups have the Efimov property we make use of the notion of *weak σ -lattice* introduced in [16].

In Section 3 we further develop a technique of strong σ -lattices of continuous mappings (or homomorphisms) which plays an important role in the subsequent Sections 4 and 5. Several of our results in Section 3 have value in their own right. For example, we show in Theorem 3.11 that if K is a compact invariant subgroup of a topological group G and the quotient group G/K has a strong σ -lattice of open homomorphisms onto topological groups with a countable network (countable base), then G also has such a lattice of open homomorphisms.

In Section 4 we consider several topological properties of extensions of the almost connected pro-Lie groups. With the help of results of Section 3 we extend the conclusions of Theorems 2.1 and 2.10 to topological groups H which contain a compact invariant subgroup K such that the quotient group H/K is *homeomorphic* to the product $C \times \prod_{i \in I} H_i$, where C is a compact group and each H_i is a topological group with a countable network (see Theorem 4.1).

Convergence properties of almost connected pro-Lie groups are considered in Section 5. We prove in Theorem 5.2 that if H is an almost connected pro-Lie group and P is a union of G_δ -sets in H , then the closure of P and the sequential closure

of P in H coincide. In other words, for every $x \in \overline{P}$, the set P contains a sequence converging to x .

1.1. Notation and terminology. A topological group G is ω -*narrow* if it can be covered by countably many translates of an arbitrary neighborhood of the identity.

A topological space X is said to be ω -*cellular* or, in symbols, $cel_\omega(X) \leq \omega$ if every family γ of G_δ -sets in X contains a countable subfamily λ such that $\bigcup \lambda$ is dense in $\bigcup \gamma$. It is clear that every ω -cellular space has countable cellularity. The class of ω -cellular spaces is considerably narrower than the class of spaces of countable cellularity—a space X of countable pseudocharacter satisfies $cel_\omega(X) \leq \omega$ if and only if it is hereditarily separable.

The union of a family of G_δ -sets in a space X is said to be a $G_{\delta,\Sigma}$ -set. We also recall that a topological space X is an *Efimov* space if the closure in X of every $G_{\delta,\Sigma}$ -set is a G_δ -set. A subset Z of X is a *zero-set* if there exists a continuous real-valued function f on X such that $Z = f^{-1}(0)$. Clearly every closed G_δ -set in a normal space is a zero-set.

The Hewitt–Nachbin completion (or *realcompactification*) of a Tychonoff space X , denoted by vX , is a realcompact space containing a dense homeomorphic copy of X , such that X is C -embedded in vX (see [7, Section 3.11]).

A Hausdorff space X is a *Lindelöf Σ -space* if X is a continuous image of a Hausdorff space Y which admits a perfect mapping onto a separable metrizable space (for basic properties of Lindelöf Σ -spaces see [2, Section 5.3]).

A topological space X is *weakly Lindelöf* if every open cover of X contains a countable subfamily which covers a dense subset of X .

A topological space X has a G_δ -*diagonal* if the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is a G_δ -set in X^2 . As usual we say that a space X is *submetrizable* if it admits a coarser metrizable topology. It is clear that every submetrizable space X is Hausdorff and has a *regular G_δ -diagonal*, i.e. the diagonal Δ_X in X^2 is the intersection of countably many of its closed neighborhoods.

All topological spaces and topological groups are assumed to be Hausdorff.

2. CONTINUOUS HOMOMORPHIC IMAGES OF PRO-LIE GROUPS

In this section we study the properties of continuous homomorphic images of *almost connected* pro-Lie groups. It is worth mentioning that even quotient groups of pro-Lie groups may fail to be pro-Lie groups [9, Corollary 4.11].

Let us recall that a *paratopological* group is a group with a topology such that multiplication on the group is continuous (but inversion can be discontinuous). The reader can consult [2, Section 2.3] for information about paratopological groups. We use this concept only in the proof of the next theorem.

Theorem 2.1. *Let a topological group H be a continuous homomorphic image of an almost connected pro-Lie group G . Then the following hold:*

- (a) *The group H is \mathbb{R} -factorizable and ω -cellular.*
- (b) *The Hewitt–Nachbin completion of H , vH , is again an \mathbb{R} -factorizable and ω -cellular topological group containing H as a topological subgroup.*

Proof. According to [10, Corollary 8.9], the group G is *homeomorphic* to the product $\mathbb{R}^\kappa \times K$, where \mathbb{R} is the real line with the usual topology, κ is a cardinal, and K is a compact topological group. Hence H is a continuous image of the topological product $\mathbb{R}^\kappa \times K$, where the factors \mathbb{R} (repeated κ times) and K are clearly Lindelöf

Σ -spaces. Therefore, combining Theorems 12 and 13 of [18] we conclude that the group H is \mathbb{R} -factorizable and ω -cellular and that the Hewitt–Nachbin completion of H is an \mathbb{R} -factorizable paratopological group containing H as a dense subgroup. Since H is dense in vH , it follows from [13, Proposition 2.2] that vH is in fact a topological group. Thus vH is an \mathbb{R} -factorizable topological group.

It is easy to see that the space vH is ω -cellular. Indeed, let γ be a family of G_δ -sets in vH . Since H is G_δ -dense in vH , the set $P \cap H$ is dense in P , for each $P \in \gamma$. Hence there exists a countable subfamily λ of γ such that $H \cap \bigcup \lambda$ is dense in $H \cap \bigcup \gamma$. Then $\bigcup \lambda$ is dense in $\bigcup \gamma$, i.e. the space vH is ω -cellular. \square

Corollary 2.2. *Every almost connected pro-Lie group H is \mathbb{R} -factorizable and ω -cellular.*

Remark 2.3. One cannot drop “almost connected” in Theorem 2.1. Indeed, there exists a prodiscrete group (that is, a complete group with a base of open subgroups) topologically isomorphic to a closed subgroup of the product of countable discrete Abelian groups which fails to be \mathbb{R} -factorizable — the group H in [17, Theorem 2.4] is as required. In particular, H is an ω -narrow pro-Lie group. Further, every G_δ -set in H is open, so H is an uncountable P -group. Hence H is not ω -cellular. In other words, all the conclusions of Theorem 2.1 are false for ω -narrow pro-Lie groups.

Our next aim in this section is to show that a continuous homomorphic image of an almost connected pro-Lie group is an Efimov space. For this purpose we make use the notion of a *weak σ -lattice* and its modifications.

Let us start with a definition of a partial preorder relation on continuous mappings of a given space.

Definition 2.4. Let $f: X \rightarrow Y$ and $g: X \rightarrow Z$ be continuous onto mappings of topological spaces. We write $f \prec g$ if there exists a continuous mapping $h: Y \rightarrow Z$ satisfying $g = h \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \downarrow h \\ & & Z \end{array}$$

Consider a family \mathcal{L} of continuous mappings of a space X elsewhere. We say that \mathcal{L} is a *weak σ -lattice* for X if the following conditions are fulfilled:

- (1) \mathcal{L} generates the original topology of X , i.e. \mathcal{L} separates points and closed subsets of X ;
- (2) every finite subfamily of \mathcal{L} has a lower bound in (\mathcal{L}, \prec) ;
- (3) for every decreasing sequence $p_0 \succ p_1 \succ p_2 \succ \dots$ in \mathcal{L} , there exists $p \in \mathcal{L}$ and a continuous one-to-one mapping $\phi: p(X) \rightarrow q(X)$ such that $q = \phi \circ p$, where q is the diagonal product of the family $\{p_n : n \in \omega\}$.

To visualize the notion of a weak σ -lattice, one can take a topological group H and consider the family of all quotient mappings $\pi_N: H \rightarrow H/N$ onto left coset spaces H/N , where N is an arbitrary closed subgroup of type G_δ in H .

We will call a subgroup N of a topological group H *admissible* (see [2, Section 5.5]) if there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity in H such that $U_{n+1}^3 \subset U_n$ for each $n \in \omega$ and $N = \bigcap_{n \in \omega} U_n$. It is easy to see that every admissible subgroup of H is closed and the intersection of countably many admissible subgroups of H is again an admissible subgroup of H .

Lemma 2.5. *Let H be a topological group and \mathcal{N} the family of admissible subgroups of H . For every $N \in \mathcal{N}$, let $\pi_N: H \rightarrow H/N$ be the quotient mapping onto the corresponding left coset space. Then the family $\mathcal{L} = \{\pi_N : N \in \mathcal{N}\}$ is a weak σ -lattice for H which consists of open mappings onto submetrizable spaces. In particular, H/N has a regular G_δ -diagonal for each $N \in \mathcal{N}$.*

Proof. The fact that H/N is submetrizable, for each $N \in \mathcal{N}$, follows from [2, Lemma 6.10.7]. Hence it suffices to verify that \mathcal{L} is a weak σ -lattice for H . To show that \mathcal{L} generates the topology of H , take an arbitrary open neighborhood U of the identity e in H . There exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of e in H such that $U_0 \subset U$ and $U_{n+1}^3 \subset U_n$ for each $n \in \omega$. Then $N = \bigcap_{n \in \omega} U_n$ is an admissible subgroup of H contained in U . The set $V = \pi_N(U_1)$ is open in H/N and satisfies $e \in \pi_N^{-1}(V) = U_1 N \subset U_1 U_1 \subset U_0 \subset U$. Since H is homogeneous, we see that the family \mathcal{L} generates the topology of H .

It is clear that every finite (even countable) subfamily of \mathcal{L} has a lower bound in (\mathcal{L}, \prec) since the intersection of every countable subfamily of \mathcal{N} is in \mathcal{N} . Finally, let $p_0 \succ p_1 \succ p_2 \succ \dots$ be a sequence in \mathcal{L} . For every $k \in \omega$, choose $N_k \in \mathcal{N}$ such that $p_k = \pi_{N_k}$ and let $N = \bigcap_{k \in \omega} N_k$. Denote by q the diagonal product of the mappings π_{N_k} with $k \in \omega$. Since $N \subset N_k$, there exists a continuous mapping $\varphi_k: H/N \rightarrow H/N_k$ satisfying $\pi_{N_k} = \varphi_k \circ \pi_N$, where $k \in \omega$. Let φ be the diagonal product of the family $\{\varphi_k : k \in \omega\}$. It is clear that φ is a continuous one-to-one mapping of H/N onto $q(H)$ which satisfies the equality $q = \varphi \circ \pi_N$. This proves that \mathcal{L} is a weak σ -lattice for H . \square

The following fact is close to [3, Lemma 3.4].

Lemma 2.6. *Let K be a closed invariant subgroup of a topological group G and $\pi: G \rightarrow G/K$ the quotient homomorphism. Then $\pi(N)$ is an admissible subgroup of G/K , for each admissible subgroup N of G .*

Proof. Consider an admissible subgroup N of G and take a sequence $\{U_n : n \in \omega\}$ of symmetric open neighborhoods of the identity e in G such that $U_{n+1}^3 \subset U_n$ for each $n \in \omega$ and $N = \bigcap_{n \in \omega} U_n$. Since the homomorphism π is open, $V_n = \pi(U_n)$ is a symmetric open neighborhood of the identity in G/K and $V_{n+1}^3 \subset V_n$ for each $n \in \omega$. Clearly $H = \bigcap_{n \in \omega} V_n$ is an admissible subgroup of G/K .

It remains to verify that $\pi(N) = H$. Indeed, take an arbitrary element $y \in H$. It follows from the definition of H that $\pi^{-1}(y) \cap U_n \neq \emptyset$ for each $n \in \omega$. Note that $\overline{U_{n+1}} \subset U_n$ since the set U_{n+1} is symmetric and $U_{n+1}^3 \subset U_n$. Hence we see that $\pi^{-1}(y) \cap \overline{U_{n+1}}$ is not empty for each $n \in \omega$. Since $p^{-1}(y) \cong K$ is compact and $\overline{U_{n+1}} \subset U_n \subset \overline{U_n}$ for each $n \in \omega$, we conclude that

$$\emptyset \neq p^{-1}(y) \cap \bigcap_{n \in \omega} \overline{U_n} = p^{-1}(y) \cap \bigcap_{n \in \omega} U_n = p^{-1}(y) \cap N.$$

Hence $y \in \pi(N) = H$, i.e. $\pi(N) = H$, as claimed. \square

Lemma 2.7. *Let X be a Tychonoff space such that the closure of every $G_{\delta, \Sigma}$ -set in X is a zero-set. Then the Hewitt-Hachbin completion of X has the same property.*

Proof. It is well known that X is G_δ -dense in vX , that is, X meets every non-empty G_δ -set in vX . Let P be a non-empty $G_{\delta, \Sigma}$ -set in vX . Then $Q = P \cap X$ is a non-empty $G_{\delta, \Sigma}$ -set in X . It follows from our assumptions on X that there exists a continuous real-valued function f on X such that $\overline{Q} = f^{-1}(0)$, where the closure of

Q is taken in X . Denote by g a continuous extension of f over vX . Again, making use of the fact that X is G_δ -dense in vX , we deduce the equality $\overline{P} = g^{-1}(0)$, where the closure of P is taken in vX . Hence \overline{P} is a zero-set, as claimed. \square

The next result follows from [16, Corollary 5.7] in the case of a Tychonoff space Y ; we extend the corresponding fact to Hausdorff spaces Y .

Lemma 2.8. *Let a Hausdorff space Y with a G_δ -diagonal be a continuous image of a product of Lindelöf Σ -spaces. Then Y has a countable network.*

Proof. Denote by f a continuous mapping of a product $X = \prod_{i \in I} X_i$ of Lindelöf Σ -spaces onto Y . Note that for every countable set $J \subset I$, the subproduct $X_J = \prod_{i \in J} X_i$ is a Lindelöf Σ -space and, hence, is Lindelöf. Since Y has a G_δ -diagonal, it follows from [6, Theorem 1] that f depends at most on countably many coordinates, so we can find a countable set $J \subset I$ and a mapping $h_J: X_J \rightarrow Y$, such that $f = h_J \circ p_J$, where $p_J: X \rightarrow X_J$ is the projection. Since the set J is countable, X_J is a Lindelöf Σ -space. It follows that Y is a continuous image of a Lindelöf Σ -space, so Y is in the class $\mathcal{L}\Sigma$ (see Definition 1 and Proposition 2 in [18]). Finally, according to [18, Lemma 5], every space in $\mathcal{L}\Sigma$ with a G_δ -diagonal has a countable network. \square

Lemma 2.9. *Let Y be an ω -cellular space with a weak σ -lattice of open mappings onto regular spaces with a countable network. Then the closure in Y of every $G_{\delta,\Sigma}$ -set is a zero-set, so Y is an Efimov space.*

Proof. Every regular space with a countable network is Lindelöf and, hence, normal. Since Y has a weak σ -lattice of continuous mappings onto normal spaces, it is Tychonoff. Let \mathcal{R} be a corresponding weak σ -lattice for Y of open mappings onto regular spaces with countable networks.

Let us call a subset F of Y \mathcal{R} -cylindric if one can find $f \in \mathcal{R}$ and a closed subset C of $p(Y)$ such that $F = p^{-1}(C)$. Clearly \mathcal{R} -cylindric subsets of Y are closed zero-sets in Y . We claim that every non-empty G_δ -set P in Y is the union of a family of \mathcal{R} -cylindric sets. Indeed, take an arbitrary point $x \in P$. Let $P = \bigcap_{n \in \omega} U_n$, where each U_n is open in Y . Since Y is Tychonoff and \mathcal{R} generates the topology of Y , we can find, for every $n \in \omega$, an element f_n of \mathcal{R} and an open set V_n in $f_n(Y)$ such that $f_n(x) \in O_n$ and $f_n^{-1}(\overline{O_n}) \subset U_n$. As \mathcal{R} is a weak σ -lattice for Y , there exists $g \in \mathcal{R}$ such that $g \prec f_n$ for each $n \in \omega$. Hence, for every $n \in \omega$, there exists a continuous mapping $h_n: g(Y) \rightarrow f_n(Y)$ satisfying $f_n = h_n \circ g$. Let $W_n = h_n^{-1}(O_n)$, where $n \in \omega$. Then $g(x) \in W_n$ and $g^{-1}(\overline{W_n}) \subset g^{-1}(h_n^{-1}(\overline{O_n})) = f_n^{-1}(\overline{O_n}) \subset U_n$. Put $C = \bigcap_{n \in \omega} \overline{W_n}$. Then C is a closed subset of $g(Y)$ containing the point $g(x)$. Since the space $g(Y)$ is regular and has a countable network, C is a zero-set in $g(Y)$. Hence $g^{-1}(C)$ is a zero-set in Y . It also follows from our definition of the sets W_n that

$$x \in g^{-1}(C) = \bigcap_{n \in \omega} g^{-1}(\overline{W_n}) \subset \bigcap_{n \in \omega} U_n \subset F.$$

This implies our claim.

Let F be a $G_{\delta,\Sigma}$ -set in Y . We have just proved that there exists a family γ of \mathcal{R} -cylindric sets in Y such that $F = \bigcup \gamma$. Since Y is ω -cellular, γ contains a countable subfamily λ such that $\bigcup \lambda$ is dense in $\bigcup \gamma$. Let $\lambda = \{F_n : n \in \omega\}$. For every $n \in \omega$, take an element $g_n \in \mathcal{R}$ and a closed subset C_n of $g_n(Y)$ such that $F_n = g_n^{-1}(C_n)$. Since \mathcal{R} is a weak σ -lattice for Y , there exists $g \in \mathcal{R}$ satisfying $g \prec g_n$ for each

$n \in \omega$. It follows from our choice of the mapping g that $\bigcup \lambda = g^{-1}(g(\bigcup \lambda))$. Since the mapping g is open and $\bigcup \lambda$ is dense in $\bigcup \gamma$, we have that

$$\overline{\bigcup \lambda} = g^{-1}g(\overline{\bigcup \lambda}) = g^{-1}\left(g(\overline{\bigcup \gamma})\right) \supseteq g^{-1}g(\bigcup \gamma) \supseteq \bigcup \lambda,$$

that is, $\bigcup \lambda$ is dense in $\bigcup \gamma$. This implies the equalities $\overline{\bigcup \lambda} = g^{-1}g(\overline{\bigcup \lambda}) = \overline{\bigcup \gamma}$. Since $g(\overline{\bigcup \lambda}) = \overline{g(\bigcup \lambda)}$ is a closed subset of $g(Y)$ (once again we apply the fact that the mapping g is open), the sets $C = \overline{g(\bigcup \lambda)}$ and the inverse image of C under g are zero-sets. This completes the proof of the lemma. \square

Now we are ready to prove the next main result of this section.

Theorem 2.10. *Let a topological group H be a continuous homomorphic image of an almost connected pro-Lie group G . Then the closure in H of every $G_{\delta,\Sigma}$ -set is a zero-set, so H is an Efimov space. The Hewitt–Nachbin completion of H is also an Efimov space.*

Proof. We recall that the group G is homeomorphic to the product $\mathbb{R}^\kappa \times K$, where \mathbb{R} is the real line with the usual topology, κ is a cardinal, and K is a compact topological group. Let \mathcal{N} be the family of all admissible subgroups of H . Then, by Lemma 2.5, the family $\{\pi_N : N \in \mathcal{N}\}$ is a weak σ -lattice for H which consists of open mappings onto Hausdorff spaces with a G_δ -diagonal. Since G is homeomorphic to the product of Lindelöf Σ -spaces and H is a continuous homomorphic image of G , it follows from Lemma 2.8 that the quotient space H/N has a countable network, for each $N \in \mathcal{N}$. We know in addition that the space H is ω -cellular. Hence Lemma 2.9 implies that the closure of every $G_{\delta,\Sigma}$ -set in H is a zero-set. Therefore H is an Efimov space.

Finally, we apply Lemma 2.7 to conclude that the closure in vH of every $G_{\delta,\Sigma}$ -set is a zero-set, so vH is also an Efimov space. \square

Corollary 2.11. *Every almost connected pro-Lie group H is an Efimov space.*

3. STRONG σ -LATTICES OF HOMOMORPHISMS

The following stronger version of the notion of weak σ -lattice plays an important role in the rest of the article. It enables us to generalize several results of Section 2 (see Theorems 4.1 and Corollary 4.2).

Definition 3.1 (See [16]). A family \mathcal{L} of continuous mappings of a space X elsewhere is said to be a *strong σ -lattice* for X if it has the following properties:

- (1) \mathcal{L} generates the topology of X ;
- (2) every finite subfamily of \mathcal{L} has a lower bound in \mathcal{L} with respect to the partial preorder \prec introduced in Definition 2.4;
- (3) for every decreasing sequence p_0, p_1, p_2, \dots in (\mathcal{L}, \prec) , the diagonal product of the family $\{p_n : n \in \omega\}$, say, p_ω belongs to \mathcal{L} , and if a sequence $\{x_n : n \in \omega\} \subset X$ has the property $p_k(x_n) = p_k(x_k)$ whenever $k < n$, then there exists $x \in X$ such that $p_n(x) = x_n$ for each $n \in \omega$. We will also denote the mapping $p_\omega \in \mathcal{L}$ by $\lim_{n \in \omega} p_n$.

It is worth mentioning that condition (3) of the above definition is equivalent to saying that the limit space of the inverse sequence $\{p_n(X), p_k^n : k < n, k, n \in \omega\}$ is naturally homeomorphic to $p_\omega(X)$, where $p_k^n = p_k \circ p_n^{-1}$ if $k < n$. A typical

example of a strong σ -lattice is the family of all projections of the product space $X = \prod_{i \in I} X_i$ onto countable subproducts.

Definition 3.2. A family \mathcal{L} of continuous mappings of a space X elsewhere has the *factorization property* if for every continuous mapping $f: X \rightarrow Y$ to a second countable Hausdorff space Y , one can find an element $p \in \mathcal{L}$ with $p \prec f$.

Let $X = \prod_{i \in I} X_i$ be a product of separable spaces. According to a theorem of Glicksberg in [8], the family of projections of X onto countable subproducts has the factorization property.

We recall that a continuous mapping $f: X \rightarrow Y$ is called *d-open* (or *nearly open*) if for every open set $U \subset X$, there exists an open set $V \subset Y$ such that $f(U)$ is a dense subset of V (see [14]).

Lemma 3.3. *Let \mathcal{L} be a weak σ -lattice of d-open, quotient mappings for a weakly Lindelöf space X . Then \mathcal{L} has the factorization property.*

Proof. Let f be a continuous real-valued function on X . For a non-empty set $A \subset X$, we put

$$\text{osc}(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}.$$

Clearly $\text{osc}(f, A)$ is finite iff f is bounded on A ; otherwise $\text{osc}(f, A)$ is defined to be ∞ .

Since f is continuous, for every $x \in X$ and every positive integer n , there exists an open neighborhood $U_n(x)$ of x in X such that $\text{osc}(f, U_n(x)) < 1/n$. Choose an element $p_{n,x} \in \mathcal{L}$ and an open set $V_n(x)$ in $p_{n,x}(X)$ such that $x \in p_{n,x}^{-1}(V_n(x)) \subset U_n(x)$. Let $O_n(x) = p_{n,x}^{-1}(V_n(x))$. It follows from $O_n(x) \subset U_n(x)$ that $\text{osc}(f, O_n(x)) < 1/n$. Since X is weakly Lindelöf we can find, for every integer $n > 0$, a countable set $B(n) \subset X$ such that $\bigcup_{y \in B(n)} O_n(y)$ is dense in X .

Since \mathcal{L} is a weak σ -lattice for X and the sets $B(n)$ are countable, there exists $p \in \mathcal{L}$ such that $p \prec p_{n,x}$ for all $n \in \mathbb{N}$ and $x \in B(n)$. We claim that $p \prec f$. First we show that if $x, y \in X$ and $p(x) = p(y)$, then $f(x) = f(y)$. Suppose for a contradiction that there exist $x, y \in X$ such that $p(x) = p(y)$ and $f(x) \neq f(y)$. Then $|f(x) - f(y)| \geq 3/n$ for some integer $n > 0$. Since the mapping p is d-open, there are open sets W_x and W_y in $p(X)$ such that $p(O_n(x))$ is dense in W_x and $p(O_n(y))$ is dense in W_y . It is clear that $W^* = W_x \cap W_y$ is an open neighborhood of the point $p(x)$. Since the set $\bigcup_{z \in B(n)} O_n(z)$ is dense in X , we can find $z \in B(n)$ such that $W^* \cap p(O_n(z)) \neq \emptyset$. It follows from $p \prec p_{n,z}$ that $O_n(z) = p^{-1}p(O_n(z))$ and $p(O_n(z))$ is open in $p(X)$. Since the sets $p(O_n(x)) \cap W^*$ and $p(O_n(y)) \cap W^*$ are dense in W^* , we see that $p(O_n(x)) \cap p(O_n(z)) \neq \emptyset$ and $p(O_n(y)) \cap p(O_n(z)) \neq \emptyset$. Therefore, $O_n(x) \cap O_n(z) \neq \emptyset$ and $O_n(y) \cap O_n(z) \neq \emptyset$. Take elements $a \in O_n(x) \cap O_n(z)$ and $b \in O_n(y) \cap O_n(z)$. Our choice of the sets $O_n(x)$, $O_n(y)$ and $O_n(z)$ implies that

$$|f(x) - f(y)| \leq |f(x) - f(a)| + |f(a) - f(b)| + |f(b) - f(y)| < 1/n + 1/n + 1/n = 3/n,$$

thus contradicting the inequality $|f(x) - f(y)| \geq 3/n$. This proves our claim.

It follows from the claim that there exists a real-valued function φ on $p(X)$ such that $f = \varphi \circ p$. Since p is quotient and f is continuous, the function φ is continuous as well. Hence $p \prec f$ and the lattice \mathcal{L} has the factorization property. \square

The next result complements Lemma 3.3 in the case when the mappings of \mathcal{L} are open.

Lemma 3.4. *Let \mathcal{L} be a weak σ -lattice of open mappings for a weakly Lindelöf space X . Then, for every continuous mapping $f: X \rightarrow Y$ to a Hausdorff space Y with a countable network, one can find $p \in \mathcal{L}$ satisfying $p \prec f$.*

Proof. The family \mathcal{L} has the factorization property by Lemma 3.3. Let $i: Y \rightarrow Z$ be a continuous one-to-one mapping onto a Hausdorff space Z of countable weight. Then $i \circ f$ is a continuous mapping of X onto Z . Since \mathcal{L} has the factorization property, we can find $p \in \mathcal{L}$ and a continuous mapping $\varphi: p(X) \rightarrow Z$ satisfying $i \circ f = \varphi \circ p$. Let $h = i^{-1} \circ \varphi$, $h: p(X) \rightarrow Y$. Since the mapping p is continuous and open, it follows from the equality $f = h \circ p$ that h is continuous. This proves the lemma. \square

Lemma 3.5. *Let a space X have a strong σ -lattice of open mappings onto weakly Lindelöf spaces. Then X is also weakly Lindelöf.*

Proof. Denote by \mathcal{L} a strong σ -lattice for X consisting of open mappings onto weakly Lindelöf spaces. Let γ be an open cover of X . Since \mathcal{L} generates the topology of X , we can assume without loss of generality that every element $U \in \gamma$ has the form $U = p^{-1}(V)$ for some $p \in \mathcal{L}$ and an open set $V \subset p(X)$.

Take an arbitrary element $p_0 \in \mathcal{L}$. Since p_0 is open and the image $X_0 = p_0(X)$ is weakly Lindelöf, we can find a countable subfamily λ_0 of γ such that $p_0(\bigcup \lambda_0)$ is dense in X_0 . Assume that for some $n \in \omega$ we have defined elements $p_0, \dots, p_n \in \mathcal{L}$ and countable subfamilies $\lambda_0, \dots, \lambda_n$ of γ satisfying the following conditions:

- (i) $p_n \prec \dots \prec p_0$;
- (ii) $\lambda_0 \subset \dots \subset \lambda_n$;
- (iii) $p_k(\bigcup \lambda_k)$ is dense in X_k for each $k \leq n$;
- (iv) if $k < n$, then $U = p_n^{-1}p_k(U)$ for each $U \in \lambda_k$.

Since λ_n is countable, there exists an element $p_{n+1} \in \mathcal{L}$ such that $p_{n+1} \prec p_n$ and $U = p_{n+1}^{-1}p_n(U)$ for each $U \in \lambda_n$. Again, the space $X_{n+1} = p_{n+1}(X)$ is weakly Lindelöf, so there exists a countable subfamily λ_{n+1} of γ such that $\lambda_n \subset \lambda_{n+1}$ and $p_{n+1}(\bigcup \lambda_{n+1})$ is dense in X_{n+1} . This finishes our construction of the sequences $\{p_n : n \in \omega\} \subset \mathcal{L}$ and $\{\lambda_n : n \in \omega\}$.

Since \mathcal{L} is a strong σ -lattice for X , the diagonal product of the family $\{p_n : n \in \omega\}$, say, p is in \mathcal{L} . Clearly $\lambda = \bigcup_{n \in \omega} \lambda_n$ is a countable subfamily of γ . We claim that $\bigcup \lambda$ is dense in X . Indeed, it follows from (iv) that $U = p^{-1}p(U)$ for each $U \in \lambda$. Hence $\bigcup \lambda = p^{-1}p(\bigcup \lambda)$. Further, conditions (i)–(iii) imply that $p(\bigcup \lambda)$ is dense in $p(X)$. Since the mapping p is open, we conclude that $\bigcup \lambda$ is dense in X . Therefore the space X is weakly Lindelöf. \square

Assume that $f: X \rightarrow Y$ is a homeomorphism and \mathcal{L}_X and \mathcal{L}_Y are (weak or strong) σ -lattices of mappings for the spaces X and Y , respectively. We say that \mathcal{L}_X and \mathcal{L}_Y are *isomorphic* if one can find a bijection $\Phi: \mathcal{L}_X \rightarrow \mathcal{L}_Y$ and a family of mappings $\{f_p : p \in \mathcal{L}_X\}$, where $f_p: p(X) \rightarrow \Phi(p)(Y)$ is a homeomorphism such that the following diagram commutes, for each $p \in \mathcal{L}_X$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p \downarrow & & \downarrow \Phi(p) \\ p(X) & \xrightarrow{f_p} & \Phi(p)(Y) \end{array}$$

If this happens, we say that the homeomorphism f is *induced* by an isomorphism of the lattices \mathcal{L}_X and \mathcal{L}_Y .

The following lemma describes a situation when a homeomorphism between two spaces with ‘good’ σ -lattices of open mappings is generated by an isomorphism of cofinal σ -sublattices of the two lattices.

Lemma 3.6. *Let f be a homeomorphism of a space X onto Y . Let also \mathcal{L}_X and \mathcal{L}_Y be a strong σ -lattice for X and a weak σ -lattice for Y , respectively. If the lattices \mathcal{L}_X and \mathcal{L}_Y consist of open mappings onto spaces with a countable network, then the homeomorphism f is induced by an isomorphism of cofinal strong σ -sublattices of \mathcal{L}_X and \mathcal{L}_Y . In particular, \mathcal{L}_Y contains a cofinal strong σ -lattice for Y with the factorization property.*

Proof. Clearly every space with a countable network is separable and, hence, weakly Lindelöf. By Lemma 3.5, the space X is weakly Lindelöf, and so is Y . Therefore Lemma 3.3 implies that the lattices \mathcal{L}_X and \mathcal{L}_Y have the factorization property.

Let $p \in \mathcal{L}_X$ and $q \in \mathcal{L}_Y$. We call the pair (p, q) *coherent* if there exists a homeomorphism $h: p(X) \rightarrow q(Y)$ satisfying $q \circ f = h \circ p$. Assume that (p_1, q_1) is another coherent pair such that $p_1 \prec p$ and $q_1 \prec q$, and let a homeomorphism $h_1: p_1(X) \rightarrow q_1(Y)$ satisfy $q_1 \circ f_1 = h_1 \circ p_1$. According to our assumptions there exist continuous mappings $\varphi: p_1(X) \rightarrow p(X)$ and $\psi: q_1(Y) \rightarrow q(Y)$ such that $p = \varphi \circ p_1$ and $q = \psi \circ q_1$. One can easily verify that the mappings h, h_1 and φ, ψ satisfy $h \circ \varphi = \psi \circ h_1$. This simple observation will be used in the sequel without mentioning.

Consider a sequence $\{(p_n, q_n) : n \in \omega\}$, where $p_n \in \mathcal{L}_X$, $q_n \in \mathcal{L}_Y$, the pair (p_n, q_n) is coherent, and $p_{n+1} \prec p_n$, $q_{n+1} \prec q_n$ for each $n \in \omega$. Since \mathcal{L}_X is a strong σ -lattice for X , the mapping $p_\omega = \lim_{n \in \omega} p_n$ is in \mathcal{L}_X . We can identify p_ω with the diagonal product of the family $\{p_n : n \in \omega\}$. Similarly, since \mathcal{L}_Y is a weak σ -lattice for Y , there exists $q = w\text{-}\lim_{n \in \omega} q_n$ in \mathcal{L}_Y . Let us show that the pair (p_ω, q) is coherent. In the sequel we put $X_n = p_n(X)$ and $Y_n = q_n(Y)$, where $n \in \omega$.

First, for every $n \in \omega$, take a homeomorphism $f_n: X_n \rightarrow Y_n$ witnessing the coherence of the pair (p_n, q_n) . Let q_ω be the diagonal product of the family $\{q_n : n \in \omega\}$. Then there exists a continuous one-to-one mapping $i: q(Y) \rightarrow q_\omega(Y)$ satisfying $i \circ q = q_\omega$. It will be shown below that i is a homeomorphism of $q(Y)$ onto Y_ω .

For all $n, k \in \omega$ with $k < n$, let also $q_k^n: Y_n \rightarrow Y_k$ be a continuous mapping satisfying $q_k = q_k^n \circ q_n$. Denote by Y_ω the inverse limit of the sequence $\{Y_n, q_n^{n+1} : n \in \omega\}$. Then $q_\omega(Y)$ is a dense subspace of Y_ω . For every $n \in \omega$, there exist continuous mappings $q_n^\omega: Y_\omega \rightarrow Y_n$ and $q_n^{n+1}: Y_{n+1} \rightarrow Y_n$ satisfying $q_n = q_n^\omega \circ q_\omega$ and $q_n^{n+1} \circ q_{n+1} = q_n$. Similarly, there exist continuous mappings $p_n^\omega: X_\omega \rightarrow X_n$ and $p_n^{n+1}: X_{n+1} \rightarrow X_n$ satisfying $p_n = p_n^\omega \circ p_\omega$ and $p_n^{n+1} \circ p_{n+1} = p_n$, where $X_\omega = p_\omega(X)$.

Let $f_\omega: X_\omega \rightarrow Y_\omega$ be a continuous mapping satisfying $q_n \circ f = f_n \circ p_n$ for each $n \in \omega$, i.e. f_ω is the *limit* of the sequence $\{f_n : n \in \omega\}$. It is easy to see that the

equality $q_\omega \circ f = f_\omega \circ p_\omega$ holds.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 p_\omega \downarrow & & \downarrow q \\
 X_\omega & \xrightarrow{g} & q(Y) \\
 f_\omega \searrow & & \downarrow i \\
 & & Y_\omega
 \end{array}
 \quad
 \begin{array}{c}
 \curvearrowright \\
 q_\omega
 \end{array}$$

We claim that $q_\omega(Y) = Y_\omega$ or, equivalently, $i(q(Y)) = Y_\omega$. Indeed, take an arbitrary point $y \in Y_\omega$. For every $n \in \omega$, let $y_n = q_n^\omega(y)$ and choose a point $x_n \in X_n$ with $f_n(x_n) = y_n$. It is easy to verify that $p_n^{n+1}(x_{n+1}) = x_n$ for each $n \in \omega$. To see this, we note that

$$f_n(p_n^{n+1}(x_{n+1})) = q_n^{n+1}(f_{n+1}(x_{n+1})) = q_n^{n+1}(y_{n+1}) = y_n,$$

so we have the equality $f_n(p_n^{n+1}(x_{n+1})) = y_n = f_n(x_n)$. Since f_n is a bijection of X_n onto Y_n , we conclude that $p_n^{n+1}(x_{n+1}) = x_n$, as claimed. As \mathcal{L}_X is a strong σ -lattice for X , there exists $x \in X$ such that $p_n(x) = x_n$ for each $n \in \omega$. Let $z = f(x)$. Then $q_n(z) = q_n(f(x)) = f_n(p_n(x)) = f_n(x_n) = y_n$ for each $n \in \omega$, whence it follows that $q_\omega(z) = y$. We have thus proved that q_ω maps Y onto Y_ω , so $i(q(Y)) = Y_\omega$. This implies that i is a continuous bijection. The equality $f_\omega \circ p_\omega = q_\omega \circ f$ also implies that f_ω maps X_ω onto Y_ω .

Notice that f_ω is a homeomorphism between X_ω and Y_ω as the limit of the homeomorphisms f_n , with $n \in \omega$. Since f, p_ω, q are continuous open surjective mappings, so is $g = i^{-1} \circ f_\omega$. The equality $f_\omega = i \circ g$ and the fact that i and f_ω are continuous bijections together imply that g is also a bijection. Hence g and i are homeomorphisms. The equality $q \circ f = g \circ p_\omega$ proves that the pair (p_ω, q) is coherent, as claimed. In other words, the lattices \mathcal{L}_X and \mathcal{L}_Y are ‘closed’ with respect to taking limits of sequences of coherent pairs of mappings.

Finally we show that

$$\mathcal{R} = \{p \times q : p \in \mathcal{L}_X, q \in \mathcal{L}_Y, (p, q) \text{ is coherent}\}$$

is a cofinal σ -sublattice of the product lattice

$$\mathcal{L}_X \times \mathcal{L}_Y = \{p \times q : p \in \mathcal{L}_X, q \in \mathcal{L}_Y\}.$$

To this end, take arbitrary elements $p_0 \in \mathcal{L}_X$ and $q_0 \in \mathcal{L}_Y$. Our aim is to find $p_\omega \in \mathcal{L}_X$, $q_\omega \in \mathcal{L}_Y$, and a homeomorphism $f_\omega : p_\omega(X) \rightarrow q_\omega(Y)$ satisfying $p_\omega \prec p_0$, $q_\omega \prec q_0$, and $f_\omega \circ p_\omega = q_\omega \circ f$. Since the family \mathcal{L}_X has the factorization property and $q_0 \circ f$ is a continuous mapping of X to the space $q_0(Y)$ with a countable network, we apply Lemma 3.4 to find $p_1 \in \mathcal{L}_X$ and a continuous mapping $\varphi_1 : p_1(X) \rightarrow q_0(Y)$ such that $p_1 \prec p_0$ and $q_0 \circ f = \varphi_1 \circ p_1$. Similarly, using the factorization property of \mathcal{L}_Y and Lemma 3.4, we find $q_1 \in \mathcal{L}_Y$ and a continuous mapping $\psi_1 : q_1(Y) \rightarrow p_1(X)$ such that $q_1 \prec q_0$ and $\psi_1 \circ q_1 \circ f = p_1$. Continuing this construction we define sequences $\{p_n : n \in \omega\} \subset \mathcal{L}_X$ and $\{q_n : n \in \omega\} \subset \mathcal{L}_Y$ such that

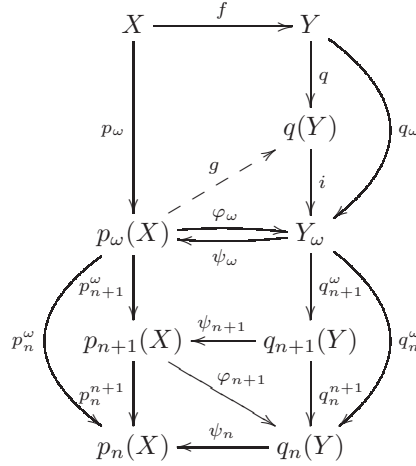
$$p_{n+1} \prec p_n, q_{n+1} \prec q_n, \text{ and } p_{n+1} \prec q_n \circ f \prec p_n$$

for each $n \in \omega$. Let $\varphi_{n+1} : p_{n+1}(X) \rightarrow q_n(Y)$ and $\psi_n : q_n(Y) \rightarrow p_n(X)$ be continuous mappings such that $\psi_n \circ q_n \circ f = p_n$ and $q_n \circ f = \varphi_{n+1} \circ p_{n+1}$. Since \mathcal{L}_X is

a strong σ -lattice for X , there exists $p_\omega = \lim_{n \in \omega} p_n$ in \mathcal{L}_X and, similarly, one can find $q = w\text{-}\lim_{n \in \omega} q_n$ in \mathcal{L}_Y . It remains to verify that the pair (p_ω, q) is coherent.

For every $n \in \omega$, $p_n^\omega = p_n \circ p_\omega^{-1}$ is a continuous open mapping of $p_\omega(X)$ onto $p_n(X)$. Denote by Y_ω the limit space of the inverse sequence $\{q_n(Y), q_n^{n+1} : n \in \omega\}$, where $q_n^{n+1} = q_{n+1}^{-1} \circ q_n$ is a continuous open mapping of $q_{n+1}(Y)$ onto $q_n(Y)$. Let $q_\omega : Y \rightarrow Y_\omega$ be the diagonal product of the family $\{q_n : n \in \omega\}$. There exists a continuous one-to-one mapping $i : q(Y) \rightarrow Y_\omega$ satisfying $q_\omega = i \circ q$. Since q_n is an onto mapping for each $n \in \omega$, the image $q_\omega(Y)$ is dense in Y_ω . For every $n \in \omega$, let $q_n^\omega : Y_\omega \rightarrow Y_n$ be a continuous mapping satisfying $q_n = q_n^\omega \circ q_\omega$.

Let $\varphi_\omega : p_\omega(X) \rightarrow Y_\omega$ be the limit of the mappings $\{\varphi_n : n \in \omega, n \geq 1\}$. Then φ_ω is continuous and satisfies the equality $\varphi_\omega \circ p_\omega = q_\omega \circ f$, so $\varphi_\omega(p_\omega(X)) = q_\omega(Y)$ is a dense subspace of Y_ω . Let also $\psi_\omega : Y_\omega \rightarrow p_\omega(X)$ be the limit of the mappings ψ_n with $n \in \omega$. Clearly the equalities $q_n^\omega \circ \varphi_\omega = \varphi_{n+1} \circ p_{n+1}^\omega$ and $p_n^\omega \circ \psi_\omega = \psi_n \circ q_n^\omega$ hold. Therefore the following diagram commutes.



Let us verify that φ_ω and ψ_ω are homeomorphisms. Making use of the commutativity of the above diagram, we obtain that

$$p_n^\omega \circ \psi_\omega \circ \varphi_\omega = \psi_n \circ q_n^\omega \circ \varphi_\omega = p_n^\omega$$

and, similarly,

$$q_n^\omega \circ \varphi_\omega \circ \psi_\omega = \varphi_{n+1} \circ p_{n+1}^\omega \circ \psi_\omega = q_n^\omega$$

for each $n \in \omega$. Since the families $\{p_n^\omega : n \in \omega\}$ and $\{q_n^\omega : n \in \omega\}$ generate the topologies of $p_\omega(X)$ and Y_ω , respectively, we conclude that $\psi_\omega \circ \varphi_\omega$ and $\varphi_\omega \circ \psi_\omega$ are identity mappings. Hence both φ_ω and ψ_ω are surjective homeomorphisms. It also follows from $q_\omega(Y) = \varphi_\omega(p_\omega(X))$ that the mapping q_ω is surjective. Since $q_\omega = i \circ q$, the one-to-one mapping i is surjective and, hence, bijective. Let $g = i^{-1} \circ \varphi_\omega$. As in the first part of the proof, we deduce from the equality $q \circ f = g \circ p_\omega$ that g is continuous. Since $\varphi_\omega = i \circ g$ is a homeomorphism of $p_\omega(X)$ onto Y_ω , it is clear that i and g are homeomorphisms. Therefore the mappings p_ω and q are coherent.

Since $p_\omega \prec p_0$ and $q \prec q_\omega \prec q_0$, we infer that the family \mathcal{R} is cofinal in $\mathcal{L}_X \times \mathcal{L}_Y$. The first part of our proof shows that \mathcal{R} is σ -closed in $\mathcal{L}_X \times \mathcal{L}_Y$. Hence the second ‘coordinates’ of the coherent pairs (p, q) form a cofinal σ -sublattice of \mathcal{L}_Y . This sublattice has the factorization property as a cofinal sublattice of \mathcal{L}_Y . \square

Lemma 3.7. *Let a topological group G be homeomorphic as a space to a product $X = \prod_{i \in I} X_i$ of spaces with a countable network. Then G has a strong σ -lattice of continuous open homomorphisms onto topological groups with a countable network.*

Proof. Let $f: X \rightarrow G$ be a homeomorphism. For every non-empty set $J \subset I$, let $p_J: X \rightarrow X_J = \prod_{i \in J} X_i$ be the projection. Then

$$\mathcal{L}_X = \{p_J : J \subset I, |J| \leq \omega\}$$

is a strong σ -lattice of open mappings for the space X . Notice that the space $X_J = p_J(X)$ has a countable network, for each countable set $J \subset I$.

Denote by \mathcal{N} the family of admissible subgroups of the group G . According to Lemma 2.5, the family

$$\mathcal{L}_G = \{\pi_N : N \in \mathcal{N}\}$$

is a weak σ -lattice for G , where $\pi_N: G \rightarrow G/N$ is a quotient mapping onto the left coset space G/N . By Lemma 2.5, the space G/N is submetrizable and, hence, has a G_δ -diagonal. Applying Lemma 2.8 we see that the space G/N has a countable network, for each $N \in \mathcal{N}$. Therefore the required conclusion follows from Lemma 3.6. \square

Problem 1. Let a topological group G be homeomorphic as a space to the product $H = \prod_{i \in I} H_i$ of Lindelöf Σ -groups. Does G have a strong σ -lattice of continuous open homomorphisms onto Lindelöf Σ -groups?

Let us recall that a space X is said to be *pseudo- ω_1 -compact* if every locally finite family of open sets in X is countable. It is clear that every weakly Lindelöf space is pseudo- ω_1 -compact, while simple examples show that the converse is false.

Lemma 3.8. *Let K be a compact subgroup of a topological group G . If the quotient space G/K is weakly Lindelöf (or pseudo- ω_1 -compact), so is G .*

Proof. Assume that G/K is weakly Lindelöf and let γ be an open cover of G . Denote by π the quotient mapping of G onto the left coset space G/K . Since each fiber of π is compact, for every $y \in G/K$ there exists a finite subfamily $\mu_y \subset \gamma$ such that $\pi^{-1}(y) \subset \bigcup \mu_y$. The mapping π is closed, so for every $y \in G/K$ we can find an open neighborhood U_y of y in G/K such that $\pi^{-1}(U_y) \subset \bigcup \mu_y$. Since the space G/K is weakly Lindelöf, the open cover $\{U_y : y \in G/K\}$ of G/K contains a countable subfamily whose union is dense in G/K . Let $\{U_y : y \in C\}$ be such a subfamily, where the set $C \subset G/K$ is countable. We claim that the union of the countable family $\mu = \bigcup_{y \in C} \mu_y \subset \gamma$ is dense in G .

Indeed, it follows from our choice of the set C that $O = \bigcup_{y \in C} U_y$ is dense in G/K . Since π is a continuous open mapping, the set $\pi^{-1}(O)$ is dense in G . It remains to note that $\pi^{-1}(O) \subset \bigcup \mu$, so the set $\bigcup \mu$ is dense in G . Since μ is a countable subfamily of γ , we conclude that G is weakly Lindelöf.

Finally, let the space G/K be pseudo- ω_1 -compact and suppose for a contradiction that γ is an uncountable locally finite family of open sets in G . Since K is compact, each coset xK in G meets at most finitely many elements of γ . Hence the family $\lambda = \{\pi(U) : U \in \gamma\}$ is also uncountable. This fact and the pseudo- ω_1 -compactness of G/K together imply that the family λ accumulates at some point $y \in G/K$. Since the quotient mapping $\pi: G \rightarrow G/K$ is perfect, the family γ must accumulate at a point $x \in G$ with $\pi(x) = y$, which is a contradiction. So the space G has to be pseudo- ω_1 -compact. \square

The following lemma is a special case of [1, Proposition 3.4]. It will be used in the proof of Theorem 3.11.

Lemma 3.9. *If C is a compact subspace of a topological group with countable pseudocharacter, then C has a countable base.*

We also need another auxiliary fact established in the proof of [2, Theorem 1.5.20]. It explains, informally speaking, to what extent one can restore the topology of a given group G in terms of a subgroup K of G and the quotient space G/K .

Lemma 3.10. *Let K be a closed subgroup of a topological group G with identity e and $\pi: G \rightarrow G/K$ be the quotient mapping onto the left coset space G/K . Assume that symmetric open neighborhoods O, V, W, W', U of e in G satisfy $V^2 \subset O$, $W \cap K \subset V$, $(W')^2 \subset W$, and $\pi(U) \subset \pi(V \cap W')$. Then $W' \cap U \subset O$.*

The next theorem is the main result of this section.

Theorem 3.11. *Let K be a compact invariant subgroup of a topological group G . If the quotient group G/K has a strong σ -lattice of open homomorphisms onto groups with a countable network (base), then G also has a strong σ -lattice of open homomorphisms onto groups with a countable network (base).*

Proof. We consider only the case when the group $H = G/K$ has a strong σ -lattice of open homomorphisms onto groups with a countable network, leaving to the reader simple modifications of the argument in the case of groups with a countable base.

Denote by \mathcal{M} a strong σ -lattice of open homomorphisms of H onto groups with a countable network. Let

$$\mathcal{L} = \{\ker g : g \in \mathcal{M}\}.$$

Since $g(H)$ is topological group with a countable network for each $g \in \mathcal{M}$, every element of \mathcal{L} is an invariant admissible subgroup of G/K . Let π be the quotient homomorphism of G onto H . Let also \mathcal{A} be the family of invariant admissible subgroups of G . Note that by Lemma 2.6, $\pi(N)$ is an admissible (and invariant) subgroup of H , for each $N \in \mathcal{A}$.

Claim A. *For every $P \in \mathcal{A}$ and every $L \in \mathcal{L}$ with $L \subset \pi(P)$, there exists $N \in \mathcal{A}$ such that $N \subset P$ and $\pi(N) = L$.*

Indeed, it suffices to put $N = P \cap \pi^{-1}(L)$. Then N is an invariant admissible subgroup of G as the intersection of the invariant admissible subgroups P and $\pi^{-1}(L)$ of G , so $N \in \mathcal{A}$. It is also clear that $\pi(N) = \pi(P \cap \pi^{-1}(L)) = \pi(P) \cap L = L$. This proves our claim.

We define a subfamily \mathcal{A}^* of \mathcal{A} by letting

$$\mathcal{A}^* = \{N \in \mathcal{A} : \pi(N) \in \mathcal{L}\}.$$

For every $N \in \mathcal{A}^*$, let $\varphi_N: G \rightarrow G/N$ be the quotient homomorphism. Let us show that $\mathcal{F} = \{\varphi_N : N \in \mathcal{A}^*\}$ is a strong σ -lattice of open homomorphisms of G onto topological groups with a countable network. This requires several steps.

Every space with a countable network is separable and, hence, weakly Lindelöf. So by Lemma 3.5, the group $H = G/K$ is weakly Lindelöf. Applying Lemma 3.8 we conclude that the group G is also weakly Lindelöf. Hence G is ω -narrow (see [2, Corollary 5.2.9]).

Step I. The group G/N has a countable network, for each $N \in \mathcal{A}^*$. Indeed, take an arbitrary element $N \in \mathcal{A}^*$. Since the subgroup K of G is compact and

invariant, NK is a closed invariant subgroup of G . Let $p: G \rightarrow G/NK$ be the quotient homomorphism. Since $N \subset NK$, there exists a continuous homomorphism $h: G/N \rightarrow G/NK$ satisfying $p = h \circ \varphi_N$. It is clear that $\varphi_N(K)$ is the kernel of h . We claim that the compact group $\varphi_N(K)$ has a countable base. First, the quotient group G/N has countable pseudocharacter because N is an admissible subgroup of G [2, Lemma 5.5.2a)]. Second, every compact subset of a topological group of countable pseudocharacter has a countable base by Lemma 3.9, whence our claim follows. Thus the group G/N contains the closed subgroup $\varphi_N(K)$ with a countable base such that the group $G/NK \cong (G/N)/\varphi_N(K) \cong (G/K)/\pi(N)$ has a countable network (notice that $\pi(N) \in \mathcal{L}$ since $N \in \mathcal{A}^*$). It now follows from [19] (the result attributed to M. Choban) that G/N has a countable network.

Step II. The family \mathcal{F} generates the original topology of G . Since the group G is ω -narrow, it follows from [2, Corollary 3.4.19] that every neighborhood of the identity in G contains an invariant admissible subgroup, i.e. an element of \mathcal{A} . We need a slightly stronger property of the family \mathcal{L} :

Claim B. *Every G_δ -set P in H with $e_H \in P$ contains an element of \mathcal{L} , where e_H is the identity of H .*

Indeed, take a countable family $\{V_n : n \in \omega\}$ of open neighborhoods of e_H in H such that $P = \bigcap_{n \in \omega} V_n$. Since \mathcal{M} is a strong σ -lattice for H , we can find, for every $n \in \omega$, an element $L_n \in \mathcal{L}$ such that $L_n \subset V_n$. Take $g_n \in \mathcal{M}$ with $\ker g_n = L_n$, where $n \in \omega$. Making use of the fact that \mathcal{M} is a strong σ -lattice for H once again, we choose $g \in \mathcal{M}$ with $g \prec g_n$ for each $n \in \omega$. Then $L = \ker g \in \mathcal{L}$ and $L \subset \bigcap_{n \in \omega} L_n \subset \bigcap_{n \in \omega} V_n = P$. This proves our claim.

Let U and V be open neighborhoods of the identity in G such that $V^2 \subset U$. It suffices to show that V contains an element N of the family \mathcal{A}^* —then the open neighborhood $\varphi_N(V)$ of the identity in $\varphi_N(G)$ satisfies $\varphi_N^{-1}\varphi_N(V) = VN \subset V^2 \subset U$. Hence we take an arbitrary element $P \in \mathcal{A}$ with $P \subset V$. Then $\pi(P)$ is an admissible subgroup of H and, hence, a G_δ -set in H . By Claim B, there exists an element $L \in \mathcal{L}$ with $L \subset \pi(P)$, so we apply Claim A to find $N \in \mathcal{A}$ such that $N \subset P$ and $\pi(N) = L$. Then $N \in \mathcal{A}^*$ and clearly $N \subset P \subset V$. We have thus proved that the family \mathcal{F} generates the original topology of G .

Step III. The family \mathcal{F} is a strong σ -lattice for G . First, take an arbitrary decreasing sequence $N_0 \supset N_1 \supset \dots \supset N_i \supset \dots$ of elements of \mathcal{A}^* . We claim that $N = \bigcap_{i \in \omega} N_i$ is an element of \mathcal{A}^* as well. Indeed, it is clear that $N \in \mathcal{A}$. It follows from the definition of the family \mathcal{A}^* that $L_i = \pi(N_i) \in \mathcal{L}$, and clearly $L_{i+1} \subset L_i$, for each $i \in \omega$. Since \mathcal{M} is a strong σ -lattice for H , we see that $L = \bigcap_{i \in \omega} L_i$ is in \mathcal{L} . Further, the compactness of the subgroup K of G implies that $\pi(N) = L$. This proves that $N \in \mathcal{A}^*$.

Let $\varphi: G \rightarrow G/N$ and $\varphi_i: G \rightarrow G/N_i$ be quotient homomorphisms, where $i \in \omega$. For every $i \in \omega$, there exists an open continuous homomorphism $p_i: G/N \rightarrow G/N_i$ satisfying $\varphi_i = p_i \circ \varphi$. For integers i, j with $0 \leq i < j$, let also $p_{j,i}: G/N_j \rightarrow G/N_i$ be an open continuous homomorphism satisfying $p_i = p_{j,i} \circ p_j$. Similarly, for every $i \in \omega$, let $h_i: G \rightarrow G/N_i K$ and $\psi_i: H \rightarrow H/L_i \cong G/N_i K$ be quotient homomorphisms satisfying $h_i = \psi_i \circ \pi$. Let also $\varpi: G/N \rightarrow H/L$ and $\varpi_i: G/N_i \rightarrow H/L_i$ be continuous homomorphisms satisfying $\varpi \circ \varphi = \psi \circ \pi$ and $\varpi_i \circ \varphi_i = \psi_i \circ \pi$, where $\psi: H \rightarrow H/L$ is the quotient homomorphism. Denote by q_i a continuous

homomorphism of H/L to H/L_i satisfying $\psi_i = q_i \circ \psi$.

$$\begin{array}{ccc}
 G & \xrightarrow{\pi} & H \\
 \varphi \downarrow & & \downarrow \psi \\
 G/N & \xrightarrow{\varpi} & H/L \\
 \varphi_i \swarrow & & \searrow \psi_i \\
 p_i \downarrow & & \downarrow q_i \\
 G/N_i & \xrightarrow{\varpi_i} & H/L_i
 \end{array}$$

It is clear that all homomorphisms in the above diagram are open.

To finish the proof of the statement in Step III (and of the theorem) it suffices to verify that the limit of the inverse sequence

$$\{G/N_k, p_{l,k} : k, l \in \omega, k < l\}$$

is topologically isomorphic to the group G/N . Let O be an arbitrary open symmetric neighborhood of the identity e_* in G/N . There exists an open symmetric neighborhood V of e_* in G/N such that $V^2 \subset O$. It is easy to find an integer $i \in \omega$ and an open symmetric neighborhood W_i of the identity in G/N_i such that $p_i^{-1}(W_i) \cap p(K) \subset V$. Indeed, $p(K)$ is a compact subspace of G/N . Let y be an arbitrary element of $p(K)$ distinct from e_* . Choose $x \in K$ with $p(x) = y$. There exists $i \in \omega$ such that $p_i(x)$ is distinct from the identity of G/N_i —otherwise $x \in \bigcap_{i \in \omega} N_i = N$ and, hence, $y = p(x) = e_*$, which contradicts our choice of y . In other words, we see that the family $\{p_k : k \in \omega\}$ of continuous homomorphisms of G/N separates points of the compact group $p(K)$. Therefore this family generates the same topology on $p(K)$ that $p(K)$ inherits from the group G/N . In particular, this implies the existence of the required $i \in \omega$ and $W_i \subset G/N_i$. We choose an open symmetric neighborhood W_{i+1} of the identity in G/N_{i+1} such that $W_{i+1}^2 \subset W_i$.

Further, choose $k \in \omega$ with $k > i$ and an open symmetric neighborhood U_k of the identity in the group H/L_k such that $q_k^{-1}(U_k) \subset \varpi(V \cap p_i^{-1}(W_{i+1}))$. This is possible since \mathcal{M} is a σ -lattice for H and $L = \bigcap_{n \in \omega} L_n$, where $L_n \in \mathcal{L}$ for each $n \in \omega$.

We claim that the open neighborhood $W^* = p_{k,i}^{-1}(W_{i+1}) \cap \varpi_k^{-1}(U_k)$ of the identity in G/N_k satisfies $p_k^{-1}(W^*) \subset O$, which shows that the family $\{p_n : n \in \omega\}$ of continuous homomorphisms of the group G/N generates the original quotient topology of G/N . Indeed, it follows from the commutativity of the above diagram that $p_k^{-1}(W^*) = p_i^{-1}(W_{i+1}) \cap \varpi^{-1}(q_k^{-1}(U_k))$. According to our choice of the sets W_i , W_{i+1} and U_k we have $W_{i+1}^2 \subset W_i$, $p_i^{-1}(W_i) \cap p(K) \subset V$, and $q_k^{-1}(U_k) \subset \varpi(V \cap p_i^{-1}(W_{i+1}))$. Therefore we can apply Lemma 3.10 with $U = \varpi^{-1}(q_k^{-1}(U_k))$, $W = p_i^{-1}(W_i)$, and $W' = p_i^{-1}(W_{i+1})$ to conclude that

$$p_k^{-1}(W^*) = p_k^{-1}(p_{k,i}^{-1}(W_{i+1})) \cap p_k^{-1}(\varpi_k^{-1}(U_k)) = p_i^{-1}(W_{i+1}) \cap \varpi^{-1}(q_k^{-1}(U_k)) \subset O.$$

Finally it remains to verify that the group G/N and the limit of the inverse sequence $S = \{G/N_k, p_{l,k} : k, l \in \omega, k < l\}$ coincide as sets. Since the family $\{p_k : k \in \omega\}$ separates points of G/N , we see that the canonical mapping of G/N to the limit of S is one-to-one. Hence it suffices to show that this mapping is onto.

Let us note that for every $i \in \omega$, the mappings in the equality $q_i \circ \varpi = \varpi_i \circ p_i$ (see the above diagram) are *bi-commutative* in the sense that $q_i^{-1}(\varpi_i(z)) = \varpi(p_i^{-1}(z))$ for each $z \in G/N_i$. Indeed, since all the mapping here are homomorphisms, we only

need to verify the latter equality in the case when z is the identity e_i of G/N_i . Direct calculations show that $q_i^{-1}(\varpi_i(e_i)) = \ker q_i = \psi(L_i)$ and $\varpi(p_i^{-1}(e_i)) = \varpi(\ker p_i) = \varpi(\varphi(N_i)) = \psi(\pi(N_i)) = \psi(L_i)$, which gives the required equality.

Let $\{x_k : k \in \omega\}$ be a sequence of points such that $x_k \in G/N_k$ and $p_{k+1,k}(x_{k+1}) = x_k$ for each $k \in \omega$. Our aim is to find $x \in G/N$ satisfying $p_k(x) = x_k$ for each k . Let $y_k = \varpi_k(x_k)$. Then $q_{k+1}(y_{k+1}) = y_k$ for all k , so our choice of the decreasing sequence $\psi_0 \succ \dots \succ \psi_n \succ \psi_{n+1} \dots \succ \psi$ in the σ -lattice \mathcal{M} implies that there exists $y \in H/L$ such that $q_k(y) = y_k$ for each $k \in \omega$. Making use of the bi-commutativity of the diagrams $q_k \circ \varpi = \varpi_k \circ p_k$, we have that $y \in q_k^{-1}(y_k) = q_k^{-1}(\varpi_k(x_k)) = \varpi(p_k^{-1}(x_k))$, for each $k \in \omega$. Hence $p_k^{-1}(x_k) \cap \varpi^{-1}(y) \neq \emptyset$ for each k . Notice that ϖ is a perfect homomorphism since its kernel is the compact group $\varphi(K)$. Hence the inclusions $p_{k+1}^{-1}(x_{k+1}) \subset p_k^{-1}(x_k)$ imply that $\varpi^{-1}(y) \cap \bigcap_{k \in \omega} p_k^{-1}(x_k) \neq \emptyset$. Take an arbitrary element $x \in \bigcap_{k \in \omega} p_k^{-1}(x_k)$. Then $p_k(x) = x_k$ for each $k \in \omega$, so the limit of the inverse sequence S and the group G/N coincide as sets. Since in addition the family $\{p_k : k \in \omega\}$ generates the quotient topology of the group G/N , we infer that G/N and the limit of S are topologically isomorphic groups. So the family \mathcal{F} is a strong σ -lattice for G .

It is worth noting that in the case when \mathcal{M} is a strong σ -lattice for G/K of open homomorphisms onto groups with countable base, our argument requires the fact that an extension of a second countable group by another second countable group is again second countable (see [2, Corollary 3.3.21]). \square

Taking $G = K$ in Theorem 3.11 (and arguing as in its proof), we obtain the following fact:

Corollary 3.12. *Let \mathcal{N} be the family of closed invariant subgroups of type G_δ in a compact topological group C . For every $N \in \mathcal{N}$, let $\pi_N : C \rightarrow C/N$ be the quotient homomorphism. Then the family $\{\pi_N : N \in \mathcal{N}\}$ is a strong σ -lattice of open homomorphisms of C onto groups with a countable base.*

The above corollary admits a slight generalization.

Corollary 3.13. *Let \mathcal{A} be the family of admissible invariant subgroups of a pseudocompact topological group P . For every $N \in \mathcal{A}$, let $\pi_N : C \rightarrow P/N$ be the quotient homomorphism. Then the family $\{\pi_N : N \in \mathcal{A}\}$ is a strong σ -lattice of open homomorphisms of P onto groups with a countable base.*

Proof. Take an arbitrary element $N \in \mathcal{A}$. Then the identity of the quotient group P/N is a G_δ -set in C/N [2, Lemma 5.5.2], so C/N is a compact metrizable group, by [5, Lemma 3.1]. Let ϱC be the Raïkov completion of C . Then ϱC is a compact topological group containing C as a dense subgroup. In fact, C meets every non-empty G_δ -set in ϱC (see [4, Theorem 1.2]). For every $N \in \mathcal{N}$, the homomorphism $\pi_N : C \rightarrow C/N$ extends to a continuous homomorphism $p_N : \varrho C \rightarrow C/N$ [2, Corollary 3.6.17]. Therefore we have the equality $p_N(C) = p_N(\varrho C) = C/N$. Since the groups ϱC and C/N are compact, the homomorphism p_N is open. To finish our argument it suffices to apply Corollary 3.12. \square

Remark 3.14. The reader surely noted the difference in our definitions of the σ -lattices \mathcal{N} and \mathcal{A} in Corollaries 3.12 and 3.13, respectively. The elements of \mathcal{N} are closed subgroups of type G_δ in the compact group C , while the elements of \mathcal{A} are *admissible* subgroups of the pseudocompact group P . Clearly every admissible

subgroup of a topological group is of type G_δ in the group, while the converse is false in the general case. However, the main reason for requiring the elements of $N \in \mathcal{A}$ to be admissible is to guarantee that the quotient group P/N have countable pseudocharacter. In fact, one can also use the family \mathcal{N} in the proof of Corollary 3.13, but this requires an extra argument.

Problem 2. Let K be a compact invariant subgroup of a topological group G such that the quotient group G/K is ω -balanced. Is the group G ω -balanced?

Problem 3. Let K be a compact invariant subgroup of a topological group G such that the quotient group G/K has a strong σ -lattice of open homomorphisms onto metrizable topological groups. Does G have a strong σ -lattice of open homomorphisms onto metrizable topological groups?

The affirmative answer to Problem 2 would imply that the answer to Problem 3 is “yes”.

Problem 4. Let a topological group G have a strong σ -lattice of open homomorphisms onto metrizable groups (groups with a countable network or a countable base) and K be a compact invariant subgroup of G . Does the group G/K have a strong σ -lattice of open homomorphisms onto metrizable groups (groups with a countable network or a countable base)?

4. EXTENSIONS OF PRO-LIE GROUPS

The class of groups having the properties described in the conclusion of Theorem 2.1 is even wider than continuous homomorphic images of almost connected pro-Lie groups. It turns out that an extension of an almost connected pro-Lie group by a compact group is in this class. This fact follows from a more general result given below.

Theorem 4.1. *Let G be a topological group and K be a compact invariant subgroup of G such that the quotient group G/K is homeomorphic as a space to the product $C \times \prod_{i \in I} H_i$, where C is a compact group and, for every $i \in I$, H_i is a topological group with a countable network. Then the group G is \mathbb{R} -factorizable, ω -cellular, and the closure of every $G_{\delta, \Sigma}$ -set in G is a zero-set, so G is an Efimov space.*

Proof. First we claim that the product group $\Pi = C \times \prod_{i \in I} H_i$ has a strong σ -lattice of open homomorphisms onto topological groups with a countable network. Indeed, Corollary 3.13 guarantees that the group C has a strong σ -lattice of open homomorphisms onto groups with a countable base. Let \mathcal{C} be such a lattice for C . For every countable set $J \subset I$, let p_J be the projection of Π onto $\prod_{i \in J} H_i$. It is easy to see that the family

$$\mathcal{M} = \{f \times p_J : f \in \mathcal{C}, \emptyset \neq J \subset I, |J| \leq \omega\}$$

is a strong σ -lattice for Π consisting of open homomorphisms onto groups with a countable network. Hence the space Π is weakly Lindelöf by Lemma 3.5. Applying Lemma 3.8 we infer that the group G is also weakly Lindelöf. Hence the group G/K is ω -narrow by [2, Corollary 5.2.9].

Denote by \mathcal{M}^* the weak σ -lattice for the group G/K which consists of all continuous open homomorphisms of G/K onto topological groups of countable pseudocharacter. Equivalently, one can define \mathcal{M}^* as the family of all quotient homomorphisms $\varphi : G/K \rightarrow (G/K)/L$, where L is an admissible invariant subgroup of

G/K . Since the spaces G/K and Π are homeomorphic, it follows from Lemma 3.6 that \mathcal{M}^* contains a cofinal subfamily, say, \mathcal{L}^* which is a strong σ -lattice for G/K .

Since the subgroup K of G is compact and invariant, we can apply Theorem 3.11 to conclude that the group G itself has a strong σ -lattice of open homomorphisms onto groups with a countable network, say, \mathcal{F} . The family \mathcal{F} has the factorization property according to Lemma 3.3. Since every topological group with a countable network is \mathbb{R} -factorizable [2, Corollary 8.1.7], we see that the group G is \mathbb{R} -factorizable as well (one can apply [2, Lemma 8.1.11] here). To complete the argument one can follow the patterns in the proofs of Theorem 2.1 and Theorem 2.10. \square

Since every almost connected pro-Lie group is *homeomorphic* to the product $C \times \mathbb{R}^\kappa$, where C is a compact group and κ is a cardinal, the next fact is immediate from Theorem 4.1.

Corollary 4.2. *If a topological group G contains a compact invariant subgroup K such that the quotient group G/K is homeomorphic to an almost connected pro-Lie group, then G is \mathbb{R} -factorizable, ω -cellular, and G is an Efimov space.*

The authors are grateful to K.H. Hofmann and S.A. Morris for providing us with an argument for the proof of the following lemma.

Lemma 4.3. *Let G be a pro-Lie group and K be a compact invariant subgroup of G such that the quotient group $H = G/K$ is a connected pro-Lie group. Then the group G is almost connected. Furthermore, if $f: G \rightarrow G/K$ is the quotient homomorphism and G_0 is the connected component of G , then $f(G_0) = H$.*

Proof. Clearly f is a closed mapping. The connected component G_0 of G is a closed invariant subgroup of G . According to [9, Corollary 4.22 (iii)], $f(G_0)$ is dense in the connected group G/K . Since G_0 is closed in G and the mapping f is closed, we see that $f(G_0) = G/K$. Hence $G_0K = G$ and the quotient group G/G_0 is compact. This implies that the group G is almost connected. \square

In the following theorem we weaken ‘connected’ to ‘almost connected’ in the assumptions of Lemma 4.3.

Proposition 4.4. *Let G be a pro-Lie group and K be a compact invariant subgroup of G such that the quotient group G/K is an almost connected pro-Lie group. Then G is almost connected.*

Proof. Let $f: G \rightarrow G/K$ be the quotient homomorphism. Denote by G_0 and H_0 the connected components of G and the quotient group $H = G/K$, respectively. Then G_0 and H_0 are closed invariant subgroups and $G^* = f^{-1}(H_0)$ is also a closed invariant subgroup of G . Hence G^* is a pro-Lie group as a closed subgroup of G [9, Theorem 3.35]. Since $G^*/K \cong H_0$ is a closed subgroup of H , we conclude that G^*/K is a connected pro-Lie group. It follows from $G_0 \leq G^* \leq G$ that the connected component of G^* is G_0 . Hence Lemma 4.3 implies that $f(G_0) = H_0$ and that $G^* = f^{-1}(H_0) = G_0K$ is an almost connected pro-Lie group. Clearly the quotient group $L = G^*/G_0$ is compact. Denote by π the quotient homomorphism of G onto G/G_0 . Then $L = G^*/G_0 \cong \pi(K)$ is a compact subgroup of G/G_0 . To finish the proof it suffices to verify that the group G/G_0 is compact, i.e. G is almost connected.

Let $p: G \rightarrow H^*$ be the quotient homomorphism, where $H^* = G/G^*$. Then we can represent p in the form $p = q \circ \pi$, where $q: G/G_0 \rightarrow (G/G_0)/L$ is the quotient homomorphism. Clearly the kernel of q is the compact subgroup L of G/G_0 . Further, the quotient group G/G^* is topologically isomorphic to the group $H/f(G_0) = H/H_0$ which is compact since H is almost connected. Hence q is a quotient homomorphism of G/G_0 onto the compact group $H^*/L \cong H/H_0$ and the kernel of q is compact. We conclude therefore that the group G/G_0 is also compact by the well-known fact that compactness is a three space property in topological groups [2, Corollary 1.5.8]. This completes the proof of the theorem. \square

We don't know if Proposition 4.4 can be generalized assuming only that the quotient group G/K is *homeomorphic* to an almost connected pro-Lie group.

It is also an open question whether Proposition 4.4 remains valid without assuming *a priori* that G is a pro-Lie group.

5. CONVERGENCE PROPERTIES OF PRO-LIE GROUPS

Our aim in this section is to show in Theorem 5.2 that the closure and sequential closure of an arbitrary $G_{\delta,\Sigma}$ -subset of an almost connected pro-Lie group coincide. As usual, we say that a subset Y of a space X is a $G_{\delta,\Sigma}$ -set in X provided that Y is the union of a family of G_δ -sets in X .

In the sequel we will use a result proved in [3, Theorem 2.8] in the case of Abelian topological groups. However, one can repeat the corresponding arguments in [3] without the use of commutativity of the groups involved there, thus obtaining the following fact:

Theorem 5.1. *Let K be a compact invariant subgroup of a topological group X and $p: X \rightarrow X/K$ the quotient homomorphism. If Y is a zero-dimensional compact subspace of X/K , then there exists a continuous mapping $s: Y \rightarrow X$ satisfying $p \circ s = Id_Y$.*

Theorem 5.2. *Let H be an almost connected pro-Lie group. Then, for every $G_{\delta,\Sigma}$ -set P in H and every point $x \in \overline{P}$, the set P contains a sequence converging to x . In other words, the closure of P and the sequential closure of P in H coincide.*

Proof. By [10, Corollary 8.9], the group H is homeomorphic to the product $\mathbb{R}^\kappa \times K$, where \mathbb{R} is the real line with the usual topology, κ is a cardinal, and K is a compact topological group. Therefore, it suffices to deduce the conclusion of the theorem in the case when H is topologically isomorphic to the product group $\mathbb{R}^\kappa \times K$. For simplicity we identify H with $\mathbb{R}^\kappa \times K$.

Let P be a $G_{\delta,\Sigma}$ -set in H and $x \in \overline{P} \setminus P$ be an arbitrary point. Take a family γ of G_δ -sets in H such that $P = \bigcup \gamma$. Refining the elements of γ , if necessary, we can assume that every element of γ is a zero-set in H . It follows from Corollary 2.2 that $cel_\omega(H) \leq \omega$, so γ contains a countable subfamily whose union is dense in the union of γ . This in turn enables us to assume that the family γ is countable.

Let \mathcal{C} be a strong σ -lattice of continuous open homomorphisms of the compact group K onto second countable topological groups. For every non-empty set $J \subset \kappa$, we denote by p_J the projection of \mathbb{R}^κ onto \mathbb{R}^J . As in the proof of Theorem 4.1 one can verify that the family

$$\mathcal{M} = \{p_J \times f : f \in \mathcal{C}, \emptyset \neq J \subset \kappa, |J| \leq \omega\}$$

is a strong σ -lattice for H and, therefore, \mathcal{M} has the factorization property. Since every element of γ is a zero-set in H , we conclude that there exists an element $\varphi \in \mathcal{M}$ such that $F = \varphi^{-1}\varphi(F)$, for each $F \in \gamma$. Let $\varphi = p_J \times f$, where J is a non-empty countable subset of κ and $f \in \mathcal{C}$. Then the groups $C = f(K)$ and $\mathbb{R}^J \times C$ are second countable and $\varphi(x) \in \overline{\varphi(\bigcup \gamma)}$. Hence we can find a sequence $\{z_n : n \in \omega\} \subset \varphi(\bigcup \gamma)$ converging to the point $\varphi(x)$.

For every $n \in \omega$, let $z_n = (u_n, c_n)$, where $u_n \in \mathbb{R}^J$ and $c_n \in C$. Let also $x = (x^*, a)$, where $x^* \in \mathbb{R}^\kappa$ and $a \in K$. Then $\varphi(x) = (y^*, b)$, where $y^* = p_J(x^*) \in \mathbb{R}^J$ and $b = f(a) \in C$. It is clear that the sequences $\{u_n : n \in \omega\}$ and $\{c_n : n \in \omega\}$ converge to y^* and b , respectively. For every $n \in \omega$, we define a point $y_n \in \mathbb{R}^\kappa$ by the rule $y_n(\alpha) = u_n(\alpha)$ if $\alpha \in J$ and $y_n(\alpha) = x^*(\alpha)$ if $\alpha \in \kappa \setminus J$. Then the sequence $\{y_n : n \in \omega\}$ converges to x^* , so the sequence $\{(y_n, c_n) : n \in \omega\}$ converges to (x^*, b) .

Let j be the identity mapping of \mathbb{R}^κ onto itself and e^* the identity element of \mathbb{R}^κ . Then $\pi = j \times f$ is a continuous homomorphism of H onto the group $\mathbb{R}^\kappa \times C$ which satisfies $\ker \pi \subset \{e^*\} \times K$ and $\pi \prec \varphi$. Since f is open, the product homomorphism π is open as well. It follows from $\pi \prec \varphi$ that $F = \pi^{-1}\pi(F)$, for each $F \in \gamma$. We claim that $(y_n, c_n) \in \pi(\bigcup \gamma)$, for each $n \in \omega$. Indeed, the homomorphism φ can be represented as the composition $\psi \circ \pi$, where $\psi : \mathbb{R}^\kappa \times C \rightarrow \mathbb{R}^J \times C$ is a homomorphism defined by $\psi(u, c) = (p_J(u), c)$ for all $u \in \mathbb{R}^\kappa$ and $c \in C$. Take an arbitrary integer $n \in \omega$. It follows from our definition of the element y_n that $\psi(y_n, c_n) = (u_n, c_n) = z_n$, so there exists $F \in \gamma$ such that $z_n \in \varphi(F)$. Since $F = \varphi^{-1}\varphi(F)$ and $\varphi = \psi \circ \pi$, we see that $(y_n, c_n) \in \pi(F)$. This proves our claim.

Let $t_n = (y_n, c_n)$, for each $n \in \omega$. The sequence $\{t_n : n \in \omega\}$ converges to the element $t = (x^*, b) = \pi(x)$ of $\mathbb{R}^\kappa \times C$. Hence $B = \{t\} \cup \{t_n : n \in \omega\}$ is a compact zero-dimensional subset of $\mathbb{R}^\kappa \times C$. Since the kernel of π , say, N is contained in $\{e^*\} \times K$, it is clear that N is a compact subgroup of H . So we can apply Theorem 5.1 to find a continuous mapping $s : B \rightarrow H$ satisfying $\pi \circ s = Id_B$. Let $x_n = s(t_n)$, for each $n \in \omega$. Then the sequence $\{x_n : n \in \omega\}$ converges to the element $h = s(t) \in H$, where $\pi(h) = \pi(s(t)) = t$. It is also clear that $\pi(x_n) = \pi(s(t_n)) = t_n$, for each $n \in \omega$. Since $F = \pi^{-1}\pi(F)$, for each $F \in \gamma$, we see that $\{x_n : n \in \omega\} \subset \bigcup \gamma$.

If $h = x$, we are done. So let $h \neq x$. It follows from $\pi(x) = t = \pi(h)$ that $g = h^{-1}x \in N = \ker \pi$. For every $n \in \omega$, we put $x'_n = x_n \cdot g$. Then the sequence $\{x'_n : n \in \omega\}$ converges to $x = h \cdot g$. Let $n \in \omega$ be arbitrary and take $F \in \gamma$ such that $x_n \in F$. Then $x'_n = x_n \cdot g \in FN = \pi^{-1}\pi(F) = F$, whence it follows that the sequence $\{x'_n : n \in \omega\}$ is covered by γ . This completes the proof of the theorem. \square

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DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER SHEVA, P.O.B. 653, ISRAEL

E-mail address: arkady@math.bgu.ac.il

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA, AV. SAN RAFAEL ATLIXCO 186, COL. VICENTINA, DEL. IZTAPALAPA, C.P. 09340, CIUDAD DE MÉXICO, MEXICO

E-mail address: mich@xanum.uam.mx